Symmetry Classification of Diatomic Molecular Chains

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April 4, 2001

Abstract

A symmetry classification of possible interactions in a diatomic molecular chain is provided. For nonlinear interactions the group of Lie point transformations, leaving the lattice invariant and taking solutions into solutions, is at most five-dimensional. An example is considered in which subgroups of the symmetry group are used to reduce the dynamical differential-difference equations to purely difference ones.

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1 Introduction

The purpose of this article is to analyze possible interactions in a long one-dimensional molecule consisting of two types of atoms. The model we consider is a very general one, described by the equations

$$\begin{aligned} E_1 &\equiv \ddot{x}_n - F_n(\xi_n, t) - G_n(\eta_{n-1}, t) = 0, \\ E_2 &\equiv \ddot{y}_n - K_n(\xi_n, t) - P_n(\eta_n, t) = 0, \end{aligned} \tag{1.1}$$

where the overdots denote time derivatives and x_n, y_n can be interpreted as the displacement of the n-th atom of type X or Y, respectively, from their equilibrium positions. We define

$$\xi_n \equiv y_n - x_n, \qquad \eta_n \equiv x_{n+1} - y_n \tag{1.2}$$

and t is time. The functions F_n, G_n, K_n and P_n are as yet unspecified smooth functions. Indeed, our aim is to classify such systems according to the Lie point symmetries that they allow, that is, to classify these functions F_n, G_n, K_n and P_n.

The assumptions built into the model are:

1. The atoms of type X and Y alternate along a fixed uniform one-dimensional chain with positions labeled by the integers n (see Figure 1).

2. Only nearest neighbor interactions are considered, i.e. the atom X_n interacts only with Y_{n-1} and Y_n and Y_n interacts only with X_n and X_{n+1} (see Figure 1).

3. The system is invariant under a uniform translation of all atoms in the molecule and also under a Galilei transformations of the chain.

4. The systems is strongly coupled, i.e. we assume
$$\frac{\partial F_n}{\partial \xi_n} \neq 0, \quad \frac{\partial G_n}{\partial \eta_{n-1}} \neq 0, \quad \frac{\partial K_n}{\partial \xi_n} \neq 0, \quad \frac{\partial P_n}{\partial \eta_n} \neq 0. \tag{1.3}$$

5. In the bulk of the article we assume that the interactions are nonlinear, i.e. at least one of the four functions F_n, G_n, K_n or P_n depends nonlinearly on the argument ξ or η, respectively. The linear case will be treated separately.

6. A discrete symmetry is built into the model. Indeed, the two equations (1.1) are permuted by the transformation

$$\begin{gathered} x_n \longrightarrow y_n, \quad y_n \longrightarrow x_{n+1}, \\ F_n \longrightarrow P_n, \quad G_n \longrightarrow K_n, \quad K_{n-1} \longrightarrow G_n, \quad P_{n-1} \longrightarrow F_n. \end{gathered} \tag{1.4}$$

Models of this type have many applications in classical mechanics, in molecular physics, or mathematical biology [1, 2, 3]. In applications, the form of the functions in eq.(1.1) are usually *a priori* fixed.

The formalism used in this article is the one called "intrinsic method" in earlier articles [4, 5]. It has already been applied to monoatomic molecular chains [6] and to a model with two species, or two types of atoms, distributed along a double chain [7].

In this approach the dependent variables x and y depend on one discrete variable n and one continuous variable t. Symmetry transformations, taking solutions into solutions, act on the variables x, y and t, not however on the lattice variable n. The Lie algebra of the symmetry group is realized by vector fields of the form

$$\hat{X} = \tau(x_n, y_n, t)\partial_t + \phi_n(x_n, y_n, t)\partial_{x_n} + \psi_n(x_n, y_n, t)\partial_{y_n}. \tag{1.5}$$

The functions τ, ϕ_n and ψ_n are determined from the requirement that the second prolongation of the vector field $\hat{X}$ should annihilate equations (1.1) on their solution surface. Explicitly we have [4, 5, 6, 7]

$$\begin{aligned} \mathrm{pr}^{(2)}\hat{X} &= \tau(t, x_n, y_n)\partial_t + \sum_{k=n-1}^{n+1} \phi_k(t, x_n, y_n)\partial_{x_k} \\ &+ \sum_{k=n-1}^{n+1} \psi_k(t, x_n, y_n)\partial_{y_k} + \phi_n^{tt}\partial_{\ddot{x}_n} + \psi_n^{tt}\partial_{\ddot{y}_n} \end{aligned} \tag{1.6}$$

with

$$\begin{aligned} \phi_n^{tt} &= D_t^2\phi_n - (D_t^2\tau)\,\dot{x}_n - 2(D_t\tau)\,\ddot{x}_n, \\ \psi_n^{tt} &= D_t^2\psi_n - (D_t^2\tau)\,\dot{y}_n - 2(D_t\tau)\,\ddot{y}_n \end{aligned} \tag{1.7}$$

(D_t is the total time derivative). In eq.(1.6) we have spelled out only those terms which act on eq.(1.1).

The use of this formalism is not obligatory. Indeed, the group transformations can also act on the lattice [8, 9, 10, 11] and generalized symmetries

can be very useful [12]. In this article we restrict ourselves to the intrinsic formalism, described above.

The present article is organized as follows. In Section 2 we establish the general form of the vector fields (1.5) that realize the symmetry algebra of eq.(1.1). We also derive the determining equations for the symmetries and introduce a “group of allowed transformations”. Allowed transformations take equations of the type (1.1) into other equations of the same type. They can change the functions F_n, G_n, K_n and P_n into other functions of the same arguments. As in previous articles, we classify equations into symmetry classes under the action of allowed transformations [6, 7, 13, 14]. We also establish that equations (1.1) are invariant under a two-dimensional Abelian group for any functions $F_n, \ldots, P_n$. Section 3 is devoted to Abelian symmetry algebras. We denote them $A_{j,k}$ where A means Abelian, j denotes the dimension and $k = 1, 2, 3, \ldots$ enumerates algebras of the same dimension. For each interaction we list only the maximal symmetry algebra. Section 4 is devoted to nilpotent symmetry algebras, denoted by $N_{j,k}$ with the same conventions as in Section 3. In Section 5 we find all solvable symmetry algebras with non-Abelian nilradicals ($SN_{j,k}$). In Section 6 those with Abelian nilradicals ($SA_{j,k}$). All nonsolvable symmetry algebras are listed in Section 7 ($NS_{j,k}$). In Sections 3 to 7 we consider only nonlinear interactions. Symmetries of the linear case are discussed in Section 8. Conclusions and some applications of the symmetries are summed up in the final Section 9.

2 Determining equations and allowed transformations

The algorithm for finding the symmetry algebra of eq.(1.1) is

$$\mathrm{pr}\hat{X}\, E_a\,|_{E_b=0} = 0, \qquad a = 1, 2, \quad b = 1, 2. \tag{2.1}$$

The coefficients of all terms of the type $\dot{x}_n^p\, \dot{y}_n^q$ must vanish independently and we find that the vector field (1.5) must actually have the form

$$\hat{X} = \tau(t)\partial_t + \left[\left(a + \frac{\dot{\tau}(t)}{2}\right) x_n + \lambda_n(t)\right] \partial_{x_n} + \left[\left(a + \frac{\dot{\tau}(t)}{2}\right) y_n + \mu_n(t)\right] \partial_{y_n}, \tag{2.2}$$

where a is a constant and $\lambda_n(t), \mu_n(t)$ and $\tau(t)$ are functions of the indicated variables. This form (2.2) is valid for any interactions F_n, G_n, K_n and P_n in eq.(1.1). Moreover, we have

$$\tau = \tau_0 + \tau_1\, t + \tau_2\, t^2, \tag{2.3}$$

where τ_0, τ_1 and τ_2 are constants.

The constants a, τ_i and the functions $\lambda_n(t)$ and $\mu_n(t)$ are subject to two further determining equations that involve the interaction functions explicitly. They are

$$\begin{aligned} &\ddot{\lambda}_n + \left(a - \tfrac{3}{2}\dot{\tau}\right)(F_n + G_n) + \left[\lambda_n - \mu_n - \left(a + \tfrac{\dot{\tau}}{2}\right)\xi_n\right] F_{n,\xi_n} \\ &+ \left[\mu_{n-1} - \lambda_n - \left(a + \tfrac{\dot{\tau}}{2}\right)\eta_{n-1}\right] G_{n,\eta_{n-1}} - \tau(F_{n,t} + G_{n,t}) = 0, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\ddot{\mu}_n + \left(a - \tfrac{3}{2}\dot{\tau}\right)(K_n + P_n) + \left[\lambda_n - \mu_n - \left(a + \tfrac{\dot{\tau}}{2}\right)\xi_n\right] K_{n,\xi_n} \\ &+ \left[\mu_n - \lambda_{n+1} - \left(a + \tfrac{\dot{\tau}}{2}\right)\eta_n\right] P_{n,\eta_n} - \tau(K_{n,t} + P_{n,t}) = 0. \end{aligned} \tag{2.5}$$

Our task is to perform a complete analysis of eq.(2.4) and (2.5). Conceptually, this is very similar to the problem considered in Ref. 7. However, the functions figuring in eq.(1.1) are less general than those of Ref. 7, hence the computations are simpler.

We shall classify the equations of type (1.1) into equivalence classes under the action of a group of "allowed transformations". These are transformations of the form

$$x_n = \Phi_n(\tilde{x}_n, \tilde{y}_n, \tilde{t}), \qquad y_n = \Psi_n(\tilde{x}_n, \tilde{y}_n, \tilde{t}), \qquad t = T(\tilde{t}) \tag{2.6}$$

that transform equations (1.1) into equations of the same form, but do not preserve the functions on the right hand side of eq.(1.1). The requirement that no first derivatives should appear and that the transformed functions $\tilde{F}_n$ and $\tilde{K}_n$ should depend only on $\tilde{\xi}_n$ and $\tilde{t}$, $\tilde{G}_n$ and $\tilde{P}_n$ only on $\tilde{t}$ and $\tilde{\eta}_{n-1}$ or $\tilde{\eta}_n$, respectively, implies that the transformations actually have the form

$$\begin{pmatrix} x_n(t) \\ y_n(t) \end{pmatrix} = q\, \dot{\tilde{t}}^{-1/2} \begin{pmatrix} \tilde{x}_n(\tilde{t}) \\ \tilde{y}_n(\tilde{t}) \end{pmatrix} + \begin{pmatrix} \alpha_n(t) \\ \beta_n(t) \end{pmatrix}, \tag{2.7}$$

$$\tilde{t} = \frac{c_1\, t + c_2}{c_3\, t + c_4}, \qquad c_1 c_4 - c_2 c_3 = 1, \qquad q \neq 0, \tag{2.8}$$

where $q, c_1, \ldots, c_4$ are constants and α_n and β_n are arbitrary functions of n and t.

The transformed system is

$$\begin{aligned}
\ddot{\tilde{x}}_n(\tilde{t}) &= \tilde{F}_n(\tilde{\xi}_n, \tilde{t}) + \tilde{G}_n(\tilde{\eta}_{n-1}, \tilde{t}), \\
\ddot{\tilde{y}}_n(\tilde{t}) &= \tilde{K}_n(\tilde{\xi}_n, \tilde{t}) + \tilde{P}_n(\tilde{\eta}_n, \tilde{t}),
\end{aligned} \tag{2.9}$$

with

$$\begin{pmatrix} \tilde{F}_n + \tilde{G}_n \\ \tilde{K}_n + \tilde{P}_n \end{pmatrix} = \frac{\dot{\tilde{t}}^{-3/2}}{q} \left[\begin{pmatrix} F_n(\xi_n, t) + G_n(\eta_{n-1}, t) \\ K_n(\xi_n, t) + P_n(\eta_n, t) \end{pmatrix} - \begin{pmatrix} \ddot{\alpha}_n(t) \\ \ddot{\beta}_n(t) \end{pmatrix} \right], \tag{2.10}$$

where

$$\xi_n = y_n - x_n = q\,\dot{\tilde{t}}^{-1/2}(\tilde{x}_n - \tilde{y}_n) + \alpha(t) - \beta(t), \tag{2.11}$$

$$\eta_n = x_{n+1} - y_n = q\,\dot{\tilde{t}}^{-1/2}(\tilde{x}_{n+1} - \tilde{y}_n) + \alpha_{n+1}(t) - \beta_n(t), \tag{2.12}$$

$$t = \frac{c_4\,\tilde{t} - c_2}{-c_3\,\tilde{t} + c_1}. \tag{2.13}$$

The vector field $\hat{X}$ of eq.(2.2) is transformed into a similar field with

$$\tilde{\tau}(\tilde{t}) = \tau(t(\tilde{t}))\,\dot{\tilde{t}}, \qquad \tilde{a} = a, \tag{2.14}$$

$$\begin{pmatrix} \tilde{\lambda}_n(\tilde{t}) \\ \tilde{\mu}_n(\tilde{t}) \end{pmatrix} = \frac{\dot{\tilde{t}}^{1/2}}{q} \left[(a + \frac{\dot{\tau}}{2}) \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} - \tau \begin{pmatrix} \dot{\alpha}_n \\ \dot{\beta}_n \end{pmatrix} + \begin{pmatrix} \lambda_n \\ \mu_n \end{pmatrix} \right]. \tag{2.15}$$

The transformed functions and constants must satisfy the same determining equations (2.4) and (2.5).

As mentioned in the Introduction, translational and Galilei invariance are built into the model. That is easy to check. Indeed $\lambda_n = \mu_n = 1$, $a = 0$, $\tau(t) = 0$ and $\lambda_n = \mu_n = t$, $a = 0$, $\tau(t) = 0$ are solutions of eq.(2.4) and (2.5) for F_n, G_n, K_n and P_n arbitrary. No other symmetries exist, unless some constraints on the interactions are imposed.

We shall use the allowed transformations to simplify the vector fields that occur. In particular the coefficient $\tau(t)$ of a given vector field can be transformed into one of the following expressions: 0, 1, t or t^2+1.

Our strategy will be to first find all Abelian symmetry algebras, then all nilpotent (non-Abelian) ones. Once these are known, we can determine all solvable ones, having the corresponding Abelian, or nilpotent ones as nilradicals [15]. Finally, all nonsolvable symmetry algebras will be determined, making use of their Levi decomposition [15].

Any symmetry algebra will contain the algebra

$$A_{2,1}: \quad \hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \qquad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}), \tag{2.16}$$

as a subalgebra. Allowed transformations leave the algebra (2.16) invariant. Any further element of the symmetry algebra can be transformed into one of the following ones

$$\hat{Y}_1 = \partial_t + a(x_n\,\partial_{x_n} + y_n\,\partial_{y_n}), \qquad a = 0, 1, \tag{2.17}$$

$$\hat{Y}_2 = t\,\partial_t + (a + \frac{1}{2})(x_n\,\partial_{x_n} + y_n\,\partial_{y_n}), \tag{2.18}$$

$$\hat{Y}_3 = (t^2+1)\,\partial_t + (a+t)(x_n\,\partial_{x_n} + y_n\,\partial_{y_n}), \tag{2.19}$$

$$\hat{Y}_4 = \lambda_n(t)(\partial_{x_n} + \partial_{y_n}), \qquad \ddot{\lambda}_n \neq 0, \quad \lambda_{n+1} \neq \lambda_n, \tag{2.20}$$

$$\hat{Y}_5 = \lambda_n(t)\,\partial_{x_n} + \lambda_{n+1}(t)\,\partial_{y_n}, \qquad \ddot{\lambda}_n \neq 0, \quad \lambda_{n+1} \neq \lambda_n. \tag{2.21}$$

The interactions that allow these additional terms can easily be determined from equations (2.4) and (2.5). Once this is done, we determine whether the considered interactions allows further symmetries. For each interaction, we shall only list the maximal symmetry algebra allowed, not lower-dimensional subalgebras.

3 Abelian symmetry algebras

The lowest dimensional maximal symmetry algebra is $A_{2,1}$ of eq.(2.16), present for any interactions in eq.(1.1). This algebra can be enlarged into a higher dimensional Abelian algebra by adding elements of the type (2.20) or (2.21).

The determining equations for a nonlinear system allow at most four commuting symmetry generators. Moreover, the three-dimensional symmetry algebras are never maximal.

Finally, we obtain two different four-dimensional Abelian symmetry algebras together with the interactions that allow them. They are

$$
\begin{aligned}
A_{4,1}\quad & \hat{X}_1 = \lambda_{1,n}(t)(\partial_{x_n} + \partial_{y_n}), \qquad \hat{X}_2 = \lambda_{2,n}(t)(\partial_{x_n} + \partial_{y_n}), \\
& \hat{X}_3 = \partial_{x_n} + \partial_{y_n}, \qquad \hat{X}_4 = t(\partial_{x_n} + \partial_{y_n}), \\
& F_n = F_n(\xi_n, t), \qquad G_n = \frac{\ddot{\lambda}_{1,n}}{\lambda_{1,n} - \lambda_{1,n-1}}\eta_{n-1}, \\
& K_n = K_n(\xi_n, t), \qquad P_n = \frac{\ddot{\lambda}_{1,n}}{\lambda_{1,n+1} - \lambda_{1,n}}\eta_n, \\
& \ddot{\lambda}_{1,n} \neq 0, \qquad \ddot{\lambda}_{2,n} \neq 0, \qquad \lambda_{1,n+1} \neq \lambda_{1,n}, \\
& \frac{\ddot{\lambda}_{2,n}}{\ddot{\lambda}_{1,n}} = \frac{\lambda_{2,n} - \lambda_{2,n-1}}{\lambda_{1,n} - \lambda_{1,n-1}} = \frac{\lambda_{2,n+1} - \lambda_{2,n}}{\lambda_{1,n+1} - \lambda_{1,n}}.
\end{aligned}
$$

$$
\begin{aligned}
A_{4,2}\quad & \hat{X}_1 = \lambda_{1,n}(t)\partial_{x_n} + \lambda_{1,n+1}(t)\partial_{y_n}, \qquad \hat{X}_2 = \lambda_{2,n}(t)\partial_{x_n} + \lambda_{2,n+1}(t)\partial_{y_n}, \\
& \hat{X}_3 = \partial_{x_n} + \partial_{y_n}, \qquad \hat{X}_4 = t(\partial_{x_n} + \partial_{y_n}), \\
& F_n = \frac{\ddot{\lambda}_{1,n}}{\lambda_{1,n+1} - \lambda_{1,n}}\xi_n, \qquad G_n = G_n(\eta_{n-1}, t), \\
& K_n = \frac{\ddot{\lambda}_{1,n+1}}{\lambda_{1,n+1} - \lambda_{1,n}}\xi_n, \qquad P_n = P_n(\eta_n, t), \\
& \ddot{\lambda}_{1,n} \neq 0, \qquad \ddot{\lambda}_{2,n} \neq 0, \qquad \lambda_{1,n+1} \neq \lambda_{1,n}, \\
& \frac{\ddot{\lambda}_{2,n}}{\ddot{\lambda}_{1,n}} = \frac{\lambda_{2,n} - \lambda_{2,n-1}}{\lambda_{1,n} - \lambda_{1,n-1}} = \frac{\lambda_{2,n+1} - \lambda_{2,n}}{\lambda_{1,n+1} - \lambda_{1,n}}.
\end{aligned}
$$

The algebras $A_{4,1}$ and $A_{4,2}$ are actually related by the discrete symmetry (1.4). Algebra $A_{4,1}$ is transformed into $A_{4,2}$ by the substitutions

$$
\begin{gathered}
F_n(\xi_n) \longrightarrow P_n(\eta_n), \qquad G_n(\eta_{n-1}) \longrightarrow K_n(\xi_n), \\
K_{n-1}(\xi_{n-1}) \longrightarrow G_n(\eta_{n-1}), \qquad P_{n-1}(\eta_{n-1}) \longrightarrow F_n(\xi_n), \\
\sigma_n(t)\,\partial_{x_n} \longrightarrow \sigma_{n+1}(t)\,\partial_{y_n}, \qquad \sigma_n(t)\,\partial_{y_n} \longrightarrow \sigma_n(t)\,\partial_{x_n}.
\end{gathered} \tag{3.1}
$$

The functions $\lambda_{1,n}(t)$ and $\lambda_{2,n}(t)$ in algebras $A_{4,1}$, $A_{4,2}$ satisfy the equations

$$
\frac{\ddot{\lambda}_{2,n}}{\ddot{\lambda}_{1,n}} = \frac{\lambda_{2,n} - \lambda_{2,n-1}}{\lambda_{1,n} - \lambda_{1,n-1}} = \frac{\lambda_{2,n+1} - \lambda_{2,n}}{\lambda_{1,n+1} - \lambda_{1,n}}. \tag{3.2}
$$

These equations can be solved and we obtain

$$
\begin{gathered}
\lambda_{1,n} = f(t)\lambda_{2,n} + g(t), \qquad \lambda_{2,n} = \frac{\gamma_n}{\dot{f}(t)^{1/2}} - \frac{1}{2\,\dot{f}(t)^{1/2}} \int_{t_0}^{t} \frac{\ddot{g}(s)}{\dot{f}(s)^{1/2}} \mathrm{d}s, \\
\dot{f}(t) \neq 0, \qquad \gamma_{n+1} \neq \gamma_n,
\end{gathered} \tag{3.3}
$$

where $f(t)$, $g(t)$ are arbitrary smooth functions of t and γ_n is an arbitrary function of n.

Notice that the quantities $\lambda_{1,n}(t)$ and $\lambda_{2,n}(t)$ (or $f(t)$, $g(t)$ and γ_n) figure explicitly in the interaction functions G_n and P_n of $A_{4,1}$, or respectively in F_n and K_n of $A_{4,2}$. The two algebras are thus indeed four-dimensional and completely specified.

4 Nilpotent non-Abelian symmetry algebras

Nilpotent Lie algebras exist for all dimensions dim $L \geq 3$. For dim $L = 3$ only one type exists, namely the Heisenberg algebra. It has a two-dimensional Abelian ideal. Maximality requires that this ideal be the algebra $A_{2,1}$ of eq.(2.16). The Heisenberg algebra is obtained by adding the operator $\hat{T} = \partial_t$. We then calculate the interaction allowing this symmetry algebra, and obtain

$$
\begin{aligned}
N_{3,1} \quad & \hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \qquad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}), \qquad \hat{T} = \partial_t, \\
& F_n = f_n(\xi_n), \qquad G_n = g_n(\eta_{n-1}), \\
& K_n = k_n(\xi_n), \qquad P_n = p_n(\eta_n).
\end{aligned}
$$

We mention that this algebra is invariant under the substitution (3.1).

Every nilpotent non-Abelian Lie algebra contains the Heisenberg algebra as a subalgebra. We can hence proceed by adding further operators to $N_{3,1}$. Moreover, they can only be added to the Abelian ideal. The determining equations (2.4), (2.5) allow us to add at most two operators. Maximality requires that we add precisely two. We thus obtain two mutually isomorphic five-dimensional nilpotent Lie algebras with four-dimensional Abelian ideals, namely

$$
\begin{aligned}
N_{5,1} \quad & \hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \qquad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}), \qquad \hat{T} = \partial_t, \\
& \hat{X}_3 = (\sigma_n + t^2)(\partial_{x_n} + \partial_{y_n}), \qquad \hat{X}_4 = (\sigma_n t + \tfrac{t^3}{3})(\partial_{x_n} + \partial_{y_n}), \\
& F_n = f_n(\xi_n), \qquad G_n = \frac{2}{\sigma_n - \sigma_{n-1}} \eta_{n-1}, \\
& K_n = k_n(\xi_n), \qquad P_n = \frac{2}{\sigma_{n+1} - \sigma_n} \eta_n, \qquad \sigma_{n+1} \neq \sigma_n.
\end{aligned}
$$

The second algebra $N_{5,2}$ is obtained from $N_{5,1}$ by the substitution (3.1). We mention that the interactions allowing the symmetry algebra $N_{5,1}$ are special cases of those allowing the Abelian algebra $A_{4,1}$. Similarly for $N_{5,2}$ and $A_{4,2}$.

5 Solvable nonnilpotent symmetry algebras with non-Abelian nilradicals

A solvable Lie algebra L always has a uniquely defined maximal nilpotent ideal, the nilradical $NR(L)$ [15]. If a solvable symmetry algebra of the system (1.1) has a non-Abelian nilradical, it must be $N_{3,1}$, $N_{5,1}$ or $N_{5,2}$ of Section 4, or a four-dimensional subalgebra of $N_{5,1}$ or $N_{5,2}$.

The determining equations (2.4) and (2.5) do not allow any extension of the four and five-dimensional nilpotent symmetry algebras to solvable ones.

The Heisenberg algebra $N_{3,1}$, on the other hand, leads to three different four-dimensional solvable symmetry algebras. The Lie algebras are given by four basis elements, $\hat{X}_1, \hat{X}_2$ and $\hat{T}$ of $N_{3,1}$ and an additional operator $\hat{Y}$. Below we list these elements $\hat{Y}$ together with the invariant interactions that allow the corresponding symmetry groups. In each case we present a matrix A defining the action of $\hat{Y}$ on the nilradical $N_{3,1}$.

$$SN_{4,1} \quad \hat{Y} = t\partial_t + (a+\tfrac{1}{2})(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = (\xi_n)^{\frac{2a-3}{2a+1}} f_n, \qquad G_n = (\eta_{n-1})^{\frac{2a-3}{2a+1}} g_n,$$

$$K_n = (\xi_n)^{\frac{2a-3}{2a+1}} k_n, \qquad P_n = (\eta_n)^{\frac{2a-3}{2a+1}} p_n,$$

$$A = \mathrm{diag}\,(a+\tfrac{1}{2}\,,\,1\,,\,a-\tfrac{1}{2}), \qquad a \neq -\tfrac{1}{2}\,,\,\tfrac{3}{2}.$$

$$SN_{4,2} \quad \hat{Y} = t\partial_t + (2x_n + t^2)\partial_{x_n} + (2y_n + t^2)\partial_{y_n},$$

$$F_n = f_n + \tfrac{1}{2}\ln(\xi_n), \qquad G_n = \tfrac{1}{2}\ln(\eta_{n-1}),$$

$$K_n = k_n + \tfrac{1}{2}\ln(\xi_n), \qquad P_n = \tfrac{1}{2}\ln(\eta_n),$$

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$SN_{4,3} \quad \hat{Y} = t\partial_t + \sigma_{1n}\partial_{x_n} + \sigma_{2n}\partial_{y_n},$$

$$F_n = f_n \exp\left(\frac{2\xi_n}{\sigma_{1,n} - \sigma_{2,n}}\right), \qquad G_n = g_n \exp\left(\frac{-2\eta_{n-1}}{\sigma_{1,n} - \sigma_{2,n-1}}\right),$$

$$K_n = k_n \exp\left(\frac{2\xi_n}{\sigma_{1,n} - \sigma_{2,n}}\right), \qquad P_n = p_n \exp\left(\frac{-2\eta_n}{\sigma_{1,n+1} - \sigma_{2,n}}\right),$$

$$A = \mathrm{diag}\,(0\,,\,1\,,\,-1), \qquad \sigma_{1,n} \neq \sigma_{2,n}, \qquad \sigma_{1,n+1} \neq \sigma_{2,n}.$$

The quantities $f_n, g_n, p_n, k_n, \sigma_{1,n}$ and $\sigma_{2,n}$ depend on n alone.

The transformation (3.1) does not lead to any new algebras or interactions. In the case of the algebra $SN_{4,3}$ we may have $\sigma_{2,n+1} = \sigma_{2,n}$. Then σ_2 can be transformed into $\sigma_{2,n} = \sigma = 0$. Similarly, for $\sigma_{2,n+1} \neq \sigma_{2,n}$, but $\sigma_{1,n+1} = \sigma_{1,n} \equiv \sigma$ we can transform into $\sigma = 0$.

6 Solvable nonnilpotent symmetry algebras with Abelian nilradicals

A large number of symmetry algebras of the system (1.1) is of this type. To identify and classify them, we use several known results on the structure of

solvable Lie algebras [15].

1. The nilradical $NR(L)$ is unique and its dimension satisfies
$$\dim \; NR(L) \geq \frac{1}{2} \; \dim \; L. \tag{6.1}$$

2. Any solvable Lie algebra L can be written as the algebraic sum of the nilradical $NR(L)$ and a complementary linear space F, i.e. $L = F \dot{+} NR(L)$.

3. The derived algebra is contained in its nilradical: $[L, L] \subseteq NR(L)$.

4. For an Abelian nilradical $\{\hat{X}_1, \ldots, \hat{X}_n\}$, the commutation relations can be written as
$$[\hat{X}_i, \hat{Y}_k] = (A_k)_{ij}\hat{X}_j, \quad [A_k, A_\ell] = 0, \quad [\hat{Y}_i, \hat{Y}_k] = c^\ell_{ik}\hat{X}_\ell, \quad [\hat{X}_i, \hat{X}_k] = 0, \tag{6.2}$$
where the elements $\hat{Y}_k$ are the nonnilpotent elements (outside the nilradical). The matrices A_k commute and are linearly nilindependent (i.e. no nontrivial linear combination of them is a nilpotent matrix). If only one element $\hat{Y}$ outside the nilradical exists, the nonnilpotent matrix A can be taken in Jordan canonical form.

In our case we can add that the Abelian nilradical must be one of the algebras found in Section 3. In principle, the nilradical could be a three-dimensional subalgebra of $A_{4,1}$, or $A_{4,2}$, containing $A_{2,1}$ as a subalgebra. However, it turns out that all choices of this type lead to symmetry algebras that are not maximal for the interactions that they allow.

The following solvable symmetry algebras occur.

6.1 $\dim \; NR(L) = 2$

The only two-dimensional nilradical that leads to solvable Lie algebras that are maximal for the obtained interaction is $A_{2,1}$. The solvable Lie algebras are always three dimensional. A basis for them consists of $\hat{X}_1$ and $\hat{X}_2$ of eq.(2.16) and an additional element $\hat{Y}$, given below. In each case we give the matrix A representing the action of $\hat{Y}$ on the nilradical.

$$SA_{3,1} \quad \hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}), \quad \hat{Y} = \partial_t + x_n\partial_{x_n} + y_n\partial_{y_n},$$

$$F_n = \xi_n f_n(\omega_n), \quad G_n = \eta_{n-1} g_n(\zeta_{n-1}),$$

$$K_n = \xi_n k_n(\omega_n), \quad P_n = \eta_n p_n(\zeta_n),$$

$$\omega_n = \xi_n \mathrm{e}^{-t}, \quad \zeta_n = \eta_n \mathrm{e}^{-t}, \quad A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

$$SA_{3,2} \quad \hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}), \quad \hat{Y} = t\partial_t + (a + \tfrac{1}{2})(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = t^{-2}\xi_n f_n(\omega_n), \quad G_n = t^{-2}\eta_{n-1} g_n(\zeta_{n-1}),$$

$$K_n = t^{-2}\xi_n k_n(\omega_n), \quad P_n = t^{-2}\eta_n p_n(\zeta_n),$$

$$\omega_n = \xi_n t^{-(a+\frac{1}{2})}, \quad \zeta_n = \eta_n t^{-(a+\frac{1}{2})}, \quad A = \mathrm{diag}\,(a - \tfrac{1}{2}\,,\, a + \tfrac{1}{2}).$$

$$SA_{3,3} \quad \hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}), \quad \hat{Y} = (t^2+1)\partial_t + (a+t)(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = (t^2+1)^{-2}\xi_n f_n(\omega_n), \quad G_n = (t^2+1)^{-2}\eta_{n-1} g_n(\zeta_{n-1}),$$

$$K_n = (t^2+1)^{-2}\xi_n k_n(\omega_n), \quad P_n = (t^2+1)^{-2}\eta_n p_n(\zeta_n),$$

$$\omega_n = \xi_n (t^2+1)^{-1/2} \exp[-a \arctan(t)], \quad \zeta_n = \eta_n (t^2+1)^{-1/2} \exp[-a \arctan(t)]$$

$$A = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}.$$

These three algebras are nonisomorphic (since the corresponding matrices A are not mutually conjugate). Each of these three cases is self conjugate under the substitution (3.1).

6.2 $\dim NR(L) = 4$

The nilradical could be three-dimensional, however the obtained solvable Lie algebra is never maximal. We only need to deal with four-dimensional Abelian ideals of the form $A_{4,1}$ and $A_{4,2}$. An extension to a solvable Lie algebra is only possible for special cases of the functions $\lambda_{1,n}(t)$ and $\lambda_{2,n}(t)$ figuring in the vector fields and interactions. Below we list all inequivalent extensions of $A_{4,1}$. There are precisely nine of them. The corresponding

extensions of $A_{4,2}$ are obtained by the substitution (3.1). The action of $\hat{Y}$ on $\{\hat{X}_1, \ldots, \hat{X}_4\}$ is represented by the matrix A.

$SA_{5,1}$

$$\hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \qquad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}),$$

$$\hat{X}_3 = \sigma_n \mathrm{e}^{t}(\partial_{x_n} + \partial_{y_n}), \qquad \hat{X}_4 = \sigma_n \mathrm{e}^{-t}(\partial_{x_n} + \partial_{y_n}),$$

$$\hat{Y} = \partial_t + a(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = \xi_n f_n(\omega_n), \qquad G_n = \frac{\sigma_n}{\sigma_n - \sigma_{n-1}}\eta_{n-1},$$

$$K_n = \xi_n k_n(\omega_n), \qquad P_n = \frac{\sigma_n}{\sigma_{n+1} - \sigma_n}\eta_n,$$

$$\omega_n = \xi_n \mathrm{e}^{-at}, \qquad \sigma_{n+1} \neq \sigma_n, \qquad A = \begin{pmatrix} a-1 & 0 & 0 & 0 \\ 0 & a+1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & -1 & a \end{pmatrix}.$$

$SA_{5,2}$

$$\hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \qquad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}),$$

$$\hat{X}_3 = \sigma_n \cos(t)(\partial_{x_n} + \partial_{y_n}), \qquad \hat{X}_4 = \sigma_n \sin(t)(\partial_{x_n} + \partial_{y_n}),$$

$$\hat{Y} = \partial_t + a(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = \xi_n f_n(\omega_n), \qquad G_n = \frac{-\sigma_n}{\sigma_n - \sigma_{n-1}}\eta_{n-1},$$

$$K_n = \xi_n k_n(\omega_n), \qquad P_n = \frac{-\sigma_n}{\sigma_{n+1} - \sigma_n}\eta_n,$$

$$\omega_n = \xi_n \mathrm{e}^{-at}, \qquad \sigma_{n+1} \neq \sigma_n, \qquad A = \begin{pmatrix} a & 1 & 0 & 0 \\ -1 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & -1 & a \end{pmatrix}.$$

$SA_{5,3}$ $\hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}),$

$$\hat{X}_3 = (\sigma_n + t^2)(\partial_{x_n} + \partial_{y_n}), \quad \hat{X}_4 = (\sigma_n t + \tfrac{t^3}{3})(\partial_{x_n} + \partial_{y_n}),$$

$$\hat{Y} = \partial_t + a(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = \xi_n f_n(\omega_n), \quad G_n = \frac{2\eta_{n-1}}{\sigma_n - \sigma_{n-1}},$$

$$K_n = \xi_n k_n(\omega_n), \quad P_n = \frac{2\eta_n}{\sigma_{n+1} - \sigma_n},$$

$$\omega_n = \xi_n \mathrm{e}^{-at}, \quad \sigma_{n+1} \neq \sigma_n, \quad A = \begin{pmatrix} a & 0 & 0 & -2 \\ -1 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & -1 & a \end{pmatrix}.$$

$SA_{5,4}$ $\hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}),$

$$\hat{X}_3 = \sigma_n t^{\alpha}(\partial_{x_n} + \partial_{y_n}), \quad \hat{X}_4 = \sigma_n t^{1-\alpha}(\partial_{x_n} + \partial_{y_n}),$$

$$\hat{Y} = t\partial_t + (a + \tfrac{1}{2})(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = t^{-2}\xi_n f_n(\omega_n), \quad G_n = \alpha(\alpha-1)t^{-2}\frac{\sigma_n}{\sigma_n - \sigma_{n-1}}\eta_{n-1},$$

$$K_n = t^{-2}\xi_n k_n(\omega_n), \quad P_n = \alpha(\alpha-1)t^{-2}\frac{\sigma_n}{\sigma_{n+1} - \sigma_n}\eta_n,$$

$$\omega_n = \xi_n t^{-(a+\frac{1}{2})}, \quad \sigma_{n+1} \neq \sigma_n, \quad \alpha \neq 0, 1\ ,$$
$$A = \mathrm{diag}\,(a - \alpha + \tfrac{1}{2}\,,\ a + \alpha - \tfrac{1}{2}\,,\ a + \tfrac{1}{2}\,,\ a - \tfrac{1}{2}).$$

$SA_{5,5}$ $\hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}),$

$$\hat{X}_3 = \sigma_n t^{1/2}\ln(t)(\partial_{x_n} + \partial_{y_n}), \quad \hat{X}_4 = \sigma_n t^{1/2}(\partial_{x_n} + \partial_{y_n}),$$

$$\hat{Y} = t\partial_t + (a + \tfrac{1}{2})(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = t^{-2}\xi_n f_n(\omega_n), \quad G_n = -\tfrac{1}{4}t^{-2}\frac{\sigma_n}{\sigma_n - \sigma_{n-1}}\eta_{n-1},$$

$$K_n = t^{-2}\xi_n k_n(\omega_n), \quad P_n = -\tfrac{1}{4}t^{-2}\frac{\sigma_n}{\sigma_{n+1} - \sigma_n}\eta_n,$$

$$\omega_n = \xi_n t^{-(a+\frac{1}{2})}, \quad \sigma_{n+1} \neq \sigma_n, \quad A = \begin{pmatrix} a & -1 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a + \frac{1}{2} & 0 \\ 0 & 0 & 0 & a - \frac{1}{2} \end{pmatrix}.$$

$SA_{5,6}$ $\quad \hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \qquad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}),$

$$\hat{X}_3 = \sigma_n t^{1/2} \cos[\ln(t)](\partial_{x_n} + \partial_{y_n}), \qquad \hat{X}_4 = \sigma_n t^{1/2} \sin[\ln(t)](\partial_{x_n} + \partial_{y_n}),$$

$$\hat{Y} = t\partial_t + (a + \tfrac{1}{2})(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = t^{-2}\xi_n f_n(\omega_n), \qquad G_n = -\tfrac{5}{4}t^{-2}\frac{\sigma_n}{\sigma_n - \sigma_{n-1}}\eta_{n-1},$$

$$K_n = t^{-2}\xi_n k_n(\omega_n), \qquad P_n = -\tfrac{5}{4}t^{-2}\frac{\sigma_n}{\sigma_{n+1} - \sigma_n}\eta_n,$$

$$\omega_n = \xi_n t^{-(a+\frac{1}{2})}, \qquad \sigma_{n+1} \neq \sigma_n, \qquad A = \begin{pmatrix} a & 1 & 0 & 0 \\ -1 & a & 0 & 0 \\ 0 & 0 & a+\frac{1}{2} & 0 \\ 0 & 0 & 0 & a-\frac{1}{2} \end{pmatrix}.$$

$SA_{5,7}$ $\quad \hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \qquad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}),$

$$\hat{X}_3 = [\sigma_n - \ln(t)](\partial_{x_n} + \partial_{y_n}), \qquad \hat{X}_4 = t[\sigma_n + \ln(t)](\partial_{x_n} + \partial_{y_n}),$$

$$\hat{Y} = t\partial_t + (a + \tfrac{1}{2})(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = t^{-2}\xi_n f_n(\omega_n), \qquad G_n = -t^{-2}\frac{\eta_{n-1}}{\sigma_n - \sigma_{n-1}},$$

$$K_n = t^{-2}\xi_n k_n(\omega_n), \qquad P_n = -t^{-2}\frac{\eta_n}{\sigma_{n+1} - \sigma_n},$$

$$\omega_n = \xi_n t^{-(a+\frac{1}{2})}, \qquad \sigma_{n+1} \neq \sigma_n, \qquad A = \begin{pmatrix} a+\frac{1}{2} & 0 & 1 & 0 \\ 0 & a-\frac{1}{2} & 0 & -1 \\ 0 & 0 & a+\frac{1}{2} & 0 \\ 0 & 0 & 0 & a-\frac{1}{2} \end{pmatrix}.$$

$SA_{5,8}$ $\hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}),$

$$\hat{X}_3 = \lambda_{1n}(t)(\partial_{x_n} + \partial_{y_n}), \quad \lambda_{1n} = \sigma_n(t^2+1)^{1/2}\exp[\alpha\arctan(t)],$$

$$\hat{X}_4 = \lambda_{2n}(t)(\partial_{x_n} + \partial_{y_n}), \quad \lambda_{2n} = \sigma_n(t^2+1)^{1/2}\exp[-\alpha\arctan(t)],$$

$$\hat{Y} = (t^2+1)\partial_t + (a+t)(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = (t^2+1)^{-2}\xi_n f_n(\omega_n), \quad G_n = (\alpha^2+1)(t^2+1)^{-2}\frac{\sigma_n}{\sigma_n - \sigma_{n-1}}\eta_{n-1},$$

$$K_n = (t^2+1)^{-2}\xi_n k_n(\omega_n), \quad P_n = (\alpha^2+1)(t^2+1)^{-2}\frac{\sigma_n}{\sigma_{n+1} - \sigma_n}\eta_n,$$

$$\omega_n = \xi_n(t^2+1)^{-1/2}\exp[-a\arctan(t)], \quad \sigma_{n+1} \neq \sigma_n, \quad \alpha \neq 0,$$

$$A = \begin{pmatrix} a-\alpha & 0 & 0 & 0 \\ 0 & a+\alpha & 0 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & -1 & a \end{pmatrix}.$$

$SA_{5,9}$ $\hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}),$

$$\hat{X}_3 = \sigma_n(t^2+1)^{1/2}(\partial_{x_n} + \partial_{y_n}), \quad \hat{X}_4 = \sigma_n(t^2+1)^{1/2}\arctan(t)(\partial_{x_n} + \partial_{y_n}),$$

$$\hat{Y} = (t^2+1)\partial_t + (a+t)(x_n\partial_{x_n} + y_n\partial_{y_n}),$$

$$F_n = (t^2+1)^{-2}\xi_n f_n(\omega_n), \quad G_n = (t^2+1)^{-2}\frac{\sigma_n}{\sigma_n - \sigma_{n-1}}\eta_{n-1},$$

$$K_n = (t^2+1)^{-2}\xi_n k_n(\omega_n), \quad P_n = (t^2+1)^{-2}\frac{\sigma_n}{\sigma_{n+1} - \sigma_n}\eta_n,$$

$$\omega_n = \xi_n(t^2+1)^{-1/2}\exp[-a\arctan(t)], \quad \sigma_{n+1} \neq \sigma_n, \quad \alpha \neq 0,$$

$$A = \begin{pmatrix} a & 0 & 0 & 0 \\ -1 & a & 0 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & -1 & a \end{pmatrix}.$$

In all cases the interaction terms G_n and P_n are specified, whereas F_n and K_n each involve an arbitrary function of one variable ω_n. The time dependence of the variable ω_n and the functions F_n and K_n depends on the form of the generator $\hat{Y}$.

After the substitution (3.1) we have altogether 18 five-dimensional Lie

algebras. No further symmetry generators can be added, at least in the nonlinear case studied so far.

7 Nonsolvable symmetry algebras

Any finite dimensional Lie algebra L that is not solvable is either semisimple, or has a nontrivial and unique Levi decomposition

$$L = S \triangleright R, \tag{7.1}$$

where S is semisimple and R is the radical, i.e. the maximal solvable ideal. The only semisimple Lie algebra that can be realized in terms of the vector fields (2.2) is actually simple, namely $sl(2,\mathbb{R})$. Up to allowed transformations the realization is unique (and given below by the operators $\hat{Y}_1$, $\hat{Y}_2$ and $\hat{Y}_3$). The determining equations (2.4) and (2.5) can be used to obtain the interaction invariant under the corresponding group $SL(2,\mathbb{R})$. Equations (1.1) will then be invariant under a five-dimensional group that contains the subalgebra $A_{2,1}$. We have:

$$\begin{aligned}
NS_{5,1} \quad & \hat{Y}_1 = \partial_t, \quad \hat{Y}_2 = t\partial_t + \tfrac{1}{2}(x_n\partial_{x_n} + y_n\partial_{y_n}), \quad \hat{Y}_3 = t^2\partial_t + t(x_n\partial_{x_n} + y_n\partial_{y_n}), \\
& \hat{X}_1 = \partial_{x_n} + \partial_{y_n}, \quad \hat{X}_2 = t(\partial_{x_n} + \partial_{y_n}), \\
& F_n = \xi_n^{-3}\, f_n, \quad G_n = \eta_{n-1}^{-3}\, g_n, \\
& K_n = \xi_n^{-3}\, k_n, \quad P_n = \eta_n^{-3}\, p_n.
\end{aligned}$$

The Lie algebra $NS_{5,1}$ is isomorphic to the special affine algebra $\mathit{saff}(2,\mathbb{R})$. This is the only maximal nonsolvable symmetry algebra that occurs.

This completes our analysis of possible symmetries of the system (1.1) with nonlinear interactions.

8 Symmetries of linear interactions

In Sections 3–7 we have excluded the case of linear interactions. Let us turn to this case now. We specify equations (1.1) to be

$$\begin{aligned}
\ddot{x}_n &= A_n(t)\,\xi_n + B_n(t)\,\eta_{n-1} + U_n(t), \\
\ddot{y}_n &= C_n(t)\,\xi_n + D_n(t)\,\eta_n + V_n(t)\,.
\end{aligned} \tag{8.1}$$

The system is still strongly coupled, i.e. the functions A_n, B_n, C_n, D_n are all nonzero. The determining equations reduce to

$$\ddot{\lambda}_n - (\mu_n - \lambda_n)A_n - (\lambda_n - \mu_{n-1})B_n + (a - \frac{3}{2}\dot{\tau})U_n - \tau\dot{U}_n = 0, \qquad (8.2)$$

$$\ddot{\mu}_n - (\mu_n - \lambda_n)C_n - (\lambda_{n+1} - \mu_n)D_n + (a - \frac{3}{2}\dot{\tau})V_n - \tau\dot{V}_n = 0, \qquad (8.3)$$

$$2\dot{\tau}A_n + \tau\dot{A}_n = 0, \qquad (8.4)$$

$$2\dot{\tau}B_n + \tau\dot{B}_n = 0, \qquad (8.5)$$

$$2\dot{\tau}C_n + \tau\dot{C}_n = 0, \qquad (8.6)$$

$$2\dot{\tau}D_n + \tau\dot{D}_n = 0, \qquad (8.7)$$

$$\tau = \tau_0 + \tau_1 + \tau_2\, t^2, \qquad (8.8)$$

since the coefficients of ξ_n, η_n, η_{n-1} and 1 vanish separately.

For $A_n(t), \ldots, D_n(t)$ generic, we obtain $\tau = 0$ and then only equations (8.2) and (8.3) (with $\tau = 0$) survive. These equations can be solved in the generic case and we obtain two types of symmetries, both just a consequence of linearity.

1. We take $a = 0$ and denote $(\lambda_{h,n}, \mu_{h,n})$ the general solution of the homogeneous equations, i.e. eq.(8.1) with $U_n = V_n = 0$. The vector field is

$$\hat{X}_h = \lambda_{h,n}(t)\,\partial_{x_n} + \mu_{h,n}(t)\,\partial_{y_n}. \qquad (8.9)$$

2. For $a \neq 0$ we choose $a = -1$ and denote some chosen particular solution of the inhomogeneous system (8.1) $(\lambda_{p,n}, \mu_{p,n})$. The vector field is

$$\hat{X}_p = [x_n - \lambda_{p,n}(t)]\partial_{x_n} + [y_n - \mu_{p,n}(t)]\partial_{y_n}\,. \qquad (8.10)$$

In particular, if we have $U_n = V_n = 0$, then we take $\lambda_{p,n} = \mu_{p,n} = 0$ in eq.(8.10).

The symmetry (8.9) only means that we can add any solution of the homogeneous equations to a solution of eq.(8.1). The symmetry (8.10) corresponds to the fact that a solution of the homogeneous system can be multiplied by a constant.

Let us now assume that a further symmetry generator exists. It is of the form (2.2) with $\tau(t)$ as in eq.(8.8). Allowed transformations can be used to transform τ into one of four cases. Let us consider them separately.

a) $\tau = 0$

No symmetries beyond the generic ones are obtained.

b) $\tau = 1$

Using allowed transformations we simplify the additional vector field into

$$\hat{T} = \partial_t + a(x_n\,\partial_{x_n} + y_n\,\partial_{y_n}). \tag{8.11}$$

The determining equations restrict the time dependence of the coefficients in eq.(8.1) and the system reduces to

$$\begin{aligned} \ddot{x}_n &= f_n\,\xi_n + g_n\,\eta_{n-1} + u_n \mathrm{e}^{at}, \\ \ddot{y}_n &= k_n\,\xi_n + p_n\,\eta_n + v_n \mathrm{e}^{at}. \end{aligned} \tag{8.12}$$

c) $\tau = t$

The additional vector field and invariant equations are reduced to

$$\hat{D} = t\,\partial_t + (a + \frac{1}{2})(x_n\,\partial_{x_n} + y_n\,\partial_{y_n}), \tag{8.13}$$

$$\begin{aligned} \ddot{x}_n &= \frac{f_n}{t^2}\xi_n + \frac{g_n}{t^2}\,\eta_{n-1} + u_n t^{a-\frac{3}{2}}, \\ \ddot{y}_n &= \frac{k_n}{t^2}\,\xi_n + \frac{p_n}{t^2}\,\eta_n + v_n t^{a-\frac{3}{2}}\,. \end{aligned} \tag{8.14}$$

d) $\tau = t^2 + 1$

The additional vector field and invariant equations are

$$\hat{C} = (t^2+1)\partial_t + (a+t)(x_n\partial_{x_n} + y_n\partial_{y_n}), \tag{8.15}$$

$$\begin{aligned} \ddot{x}_n &= \frac{f_n}{(t^2+1)^2}\xi_n + \frac{g_n}{(t^2+1)^2}\eta_{n-1} + \frac{u_n}{(t^2+1)^{3/2}}\exp[a\arctan(t)], \\ \ddot{y}_n &= \frac{k_n}{(t^2+1)^2}\xi_n + \frac{p_n}{(t^2+1)^2}\eta_n + \frac{v_n}{(t^2+1)^{3/2}}\exp[a\arctan(t)]\,. \end{aligned} \tag{8.16}$$

In all cases f_n, g_n, k_n, p_n, u_n and v_n are independent of t. No further symmetries exist for any of the interactions (8.12), (8.14) or (8.16).

9 Conclusions

Let us sum up the results obtained above.

For nonlinear interactions the symmetry algebra is at most five-dimensional. The following cases occur.

1. The nonsolvable algebra $NS_{5,1}$ of Section 7. The dependence of the right hand side of eq.(1.1) on ξ_n and η_n is completely specified by an inverse cube relation. The dependence on the discrete variable n remains arbitrary. The interactions are time independent.

2. The solvable Lie algebras with Abelian nilradicals $SA_{5,1}, \ldots, SA_{5,9}$ (and $SA_{5,10}, \ldots, SA_{5,18}$ by the substitution (3.1)) of Section 6.2. The interactions are all "semilinear". By this we mean that the dependence on one variable η_n is specified to be linear, whereas the dependence on ξ_n remains arbitrary (and vice versa for $SA_{5,10}, \ldots, SA_{5,18}$). The time dependence of the nonlinear terms in the interaction depends crucially on the form of the nonnilpotent element $\hat{Y}$.

   Any attempt to enlarge the symmetry algebra by further elements leads to linear interactions.

3. The nilpotent five-dimensional Lie algebras $N_{5,1}$ and the related algebra $N_{5,2}$ of Section 4. For $N_{5,1}$ the interaction is again semilinear with

G_n and P_n linear in η_{n-1} and η_n, respectively, and F_n and K_n arbitrary functions of ξ_n (and vice versa for $N_{5,2}$). The interaction is time independent.

4. Four-dimensional maximal symmetry algebras are either Abelian, or solvable with the Heisenberg algebra as a nilradical. For $A_{4,1}$ and $A_{4,2}$ the interaction is semilinear with an arbitrary time dependence in the nonlinear terms. For $SN_{4,1}$, $SN_{4,2}$ and $SN_{4,3}$ the dependence on ξ_n and η_n is completely specified as being monomial, logarithmic or exponential, respectively. There is no time dependence.

5. A three-dimensional maximal symmetry algebra is either nilpotent, or solvable with an Abelian nilradical. For $N_{3,1}$, the Heisenberg algebra, the interaction is time independent, otherwise arbitrary. The model, studied by Campa *et al* [1], namely

$$\begin{array}{rclcrcl} F_n(\xi_n) & = & \frac{1}{M_1}(k_1\xi_n+\varepsilon\beta_1\xi_n^2) & , & K_n(\xi_n) & = & -\frac{M_1}{M_2}F_n(\xi_n), \\ G_n(\eta_{n-1}) & = & -\frac{1}{M_1}(k_2\eta_{n-1}+\varepsilon\beta_2\eta_{n-1}^2) & , & P_n(\eta_n) & = & -\frac{M_1}{M_2}G_{n+1}(\eta_{n+1}), \end{array}$$

is of this type. For $SA_{3,1}$, $SA_{3,2}$ and $SA_{3,3}$ the interactions involve four arbitrary functions of one variable. The interaction is entirely specified by the element $\hat{Y}$.

6. As mentioned above, the general interaction in eq.(1.1) is invariant under the group of global translations and Galilei transformations, corresponding to the algebra $A_{2,1}$ of eq.(2.16).

The symmetries found in this article can be used to perform symmetry reduction on one hand, and to obtain new solutions from known ones, on the other.

Let us look at the example of algebra $NS_{5,1}$. The system (1.1) in this case reduces to

$$\ddot{x}_n = \frac{f_n}{\xi_n^3} + \frac{g_n}{\eta_{n-1}^3}, \qquad \ddot{y}_n = \frac{k_n}{\xi_n^3} + \frac{p_n}{\eta_n^3}. \tag{9.1}$$

The algebra $sl(2,\mathbb{R})$ has three inequivalent one-dimensional subalgebras, namely $\hat{Y}_1$, $\hat{Y}_2$ and $\hat{Y}_3 + \hat{Y}_1$. Each of them can be used to reduce the system (9.1) to a system of two difference equations. Let us look at the three individual cases separately.

Subalgebra $\hat{Y}_1$

This algebra leads to stationary solutions. We have

$$x_n = x_{n,0}, \qquad y_n = y_{n,0} \tag{9.2}$$

and hence

$$\xi_{n,0} = \left(-\frac{f_n}{g_n}\right)^{1/3} \eta_{n-1,0} = \left(-\frac{k_n}{p_n}\right)^{1/3} \eta_{n,0}. \tag{9.3}$$

Subalgebra $\hat{Y}_2$

The reduction formulas in this case are

$$x_n = x_{n,0}\sqrt{t}, \qquad y_n = y_{n,0}\sqrt{t} \tag{9.4}$$

and the recursion relations are

$$-\frac{x_{n,0}}{4} = \frac{f_n}{\xi_{n,0}^3} + \frac{g_n}{\eta_{n-1,0}^3}, \qquad -\frac{y_{n,0}}{4} = \frac{k_n}{\xi_{n,0}^3} + \frac{p_n}{\eta_{n,0}^3}. \tag{9.5}$$

Subalgebra $\hat{Y}_3 + \hat{Y}_1$

We put

$$x_n = x_{n,0}\sqrt{t^2+1}, \qquad y_n = y_{n,0}\sqrt{t^2+1} \tag{9.6}$$

and obtain the recursion relations

$$x_{n,0} = \frac{f_n}{\xi_{n,0}^3} + \frac{g_n}{\eta_{n-1,0}^3}, \qquad y_{n,0} = \frac{k_n}{\xi_{n,0}^3} + \frac{p_n}{\eta_{n,0}^3}. \tag{9.7}$$

In all three cases we can express ξ_n in terms of η_n and obtain a two term recursion relation for η_n. These can be solved, but we will not go into the details here.

Acknowledgments

We thank professor D. Levi for interesting discussions. The research of S.L. was successively supported by a FCAR Doctoral Scholarship and by

an NSERC Postdoctoral Fellowship. The research of P.W. was partly supported by NSERC of Canada and FCAR du Québec. The work reported here was started while all three authors were visiting the CIF in Cuernavaca, Mexico. We thank T. Seligman for his hospitality there. The final version was written while P.W. was visiting the Faculty of Nuclear Sciences and Physical Engineering of the Czech Technical University in Prague. He thanks Professors M. Havlicek and J. Tolar for their hospitality.

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Figure 1. Interactions between atoms of type X and Y along a molecular chain.